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Instanton Determinant with Arbitrary Quark Mass: WKB Phase-shift Method and Derivative Expansion

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Abstract

The fermion determinant in an instanton background for a quark field of arbitrary mass is studied using the Schwinger proper-time representation with WKB scattering phase shifts for the relevant partial-wave differential operators. Previously, results have been obtained only for the extreme small and large quark mass limits, not for intermediate interpolating mass values. We show that consistent renormalization and large-mass asymptotics requires up to third-order in the WKB approximation. This procedure leads to an almost analytic answer, requiring only modest numerical approximation, and yields excellent agreement with the well-known extreme small and large mass limits. We estimate that it differs from the exact answer by no more than 6% for generic mass values. In the philosophy of the derivative expansion the same amplitude is then studied using a Heisenberg-Euler-type effective action, and the leading order approximation gives a surprisingly accurate answer for all masses.

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I. INTRODUCTION

The one-loop effective action plays a central role in quantum field theory. For a specific background field it represents quantum effects of direct physical relevance (e.g., the induced vacuum energy in the given background), while for general background fields it corresponds to the proper-vertex generating functional of the theory. The ultraviolet divergence or renormalization problem for the one-loop effective action is well-understood, but it is notoriously difficult to evaluate explicitly its full finite part in the presence of a given nontrivial background. For nonabelian theories in three spatial dimensions, there are essentially only two types of exact calculations for this *full* amplitude – the Heisenberg-Euler-type nonlinear action describing vacuum polarization and pair creation effects in the background of (covariantly) constant gauge field strengths [1, 2, 3, 4], and the QCD one-loop effective action for massless quarks in a self-dual Yang-Mills background field (describing instantons) [5, 6].

To study instanton-related physics it is of fundamental importance to determine the one-loop tunnelling amplitude given by the Euclidean one-loop effective action in the background of a single (anti-)instanton [7]. 't Hooft [5] succeeded in calculating the corresponding contribution by massless scalar or quark fields exactly in the 1970s, but such an exact calculation is no longer possible if the fields have non-zero mass. For various phenomenological applications [8] (and also for the extrapolation of lattice results [9], obtained at unphysically large quark masses, to lower physical masses), it is important to have more definite information about the contributions due to quarks of not-so-small mass. In this work, we will describe our approach to this problem and present explicit numerical results.

Previously, two of us (with Kwon) [10] studied the corresponding one-loop effective action with arbitrary mass, using a smooth interpolation between the results obtained in the large and small mass limits. The expression in the large mass limit is naturally obtained from the Schwinger-DeWitt expansion [11, 12, 13] (or the heat-kernel expansion) within the proper-time representation of the effective action, while the result in the small mass limit follows from the works of 't Hooft [5], Carlitz and Creamer [14], and Kwon et al. [10]. In this work, we use a systematic approximation for any quark mass – the effective action for any mass is calculated approximately without invoking an ad hoc interpolation procedure. The basic strategy is as follows. We use the proper-time representation of the effective action, which

requires the explicit functional form of

$$F(s) = \int d^4x \ \mathrm{tr}\langle x|e^{-s(-\mathrm{D}^2)} - e^{-s(-\partial^2)}|x\rangle \tag{1.1}$$

The instanton background enters through the covariant derivative D_μ in $D^2 \equiv D_\mu D_\mu$, and this function $F(s)$ can be expressed in terms of scattering phase shifts [5, 15, 16] for the partial-wave quadratic differential operators related to D^2. We evaluate these phase shifts in the quantum-mechanical WKB approximation, extended to the third-order correction terms [17, 18] to ensure the correct small-s behavior for $F(s)$. This leads to a numerical expression for $F(s)$, which in turn yields the effective action for arbitrary mass value. The resulting mass dependence is fully consistent with the conjecture made by Kwon et al [10], on the basis of the explicitly known results at the opposite ends.

Our partial-wave WKB phase-shift method should provide a practical approximation scheme for the one-loop effective action in a broad class of background fields. In this approach, $F(s)$ generally takes a *local* form in the potentials and their derivatives – see (3.11) below. In the light of this observation, we conclude this paper by comparing to another method, the derivative expansion, which produces such local expressions. Surprisingly, the leading order of the derivative expansion provides a remarkably good approximation for general values of the quark mass.

II. EFFECTIVE ACTION, PROPER-TIME REPRESENTATION, AND PHASE SHIFTS

Due to a hidden supersymmetry of the system with a quark in a background instanton field, the one-loop effective action of a Dirac spinor field of mass m (and isospin $\frac{1}{2}$), $\Gamma^F(A;m)$, can be related to the corresponding scalar effective action (for a complex scalar of mass m and isospin $\frac{1}{2}$) by [5, 6, 10]

$$\Gamma^F(A;m) = -\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right) - 2\,\Gamma^S(A;m), \tag{2.1}$$

The first contribution corresponds to the existence of a zero eigenvalue in the spectrum of the Dirac operator for a single instanton background. This relationship (which is special to a self-dual background) has the important consequence that it is sufficient to consider the scalar effective action $\Gamma^S(A;m)$ to learn also about the corresponding fermion effective $\Gamma^F(A;m)$, for any mass value m.

We consider an SU(2) single instanton background [5]

$$A_\mu(x) \equiv A_\mu^a(x)\frac{\tau^a}{2} = \frac{\eta_{\mu\nu a}\tau^a x_\nu}{r^2+\rho^2}, \quad (\mu = 1,2,3,4;\ r \equiv \sqrt{x_\mu x_\mu}) \tag{2.2}$$

$$F_{\mu\nu}(x) \equiv F_{\mu\nu}^a(x)\frac{\tau^a}{2} = -\frac{2\rho^2\eta_{\mu\nu a}\tau^a}{(r^2+\rho^2)^2}, \tag{2.3}$$

The regularized one-loop scalar effective action has the proper-time representation

$$\begin{aligned}\Gamma_\Lambda^S(A;m) &= -\int_0^\infty \frac{ds}{s}(e^{-m^2 s} - e^{-\Lambda^2 s})\int d^4x\ \mathrm{tr}\langle x|e^{-s(-\mathrm{D}^2)} - e^{-s(-\partial^2)}|x\rangle \\ &\equiv -\int_0^\infty \frac{ds}{s}(e^{-m^2 s} - e^{-\Lambda^2 s})F(s),\end{aligned} \tag{2.4}$$

where $D^2 \equiv D_\mu D_\mu$ with $D_\mu = \partial_\mu - iA_\mu(x)$. From this one obtains the renormalized effective action, in the minimal subtraction scheme, as

$$\begin{aligned}\Gamma^S(A;m) &= \lim_{\Lambda\to\infty}\left[\Gamma_\Lambda^S(A;m) - \frac{1}{12}\frac{1}{(4\pi)^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\ \mathrm{tr}(F_{\mu\nu}F_{\mu\nu})\right] \\ &\equiv \lim_{\Lambda\to\infty}\left[\Gamma_\Lambda^S(A;m) - \frac{1}{12}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right].\end{aligned} \tag{2.5}$$

Moreover, by dimensional considerations, we may introduce the modified scalar effective action $\tilde{\Gamma}^S(m\rho)$, which is a function of $m\rho$ only, defined by

$$\Gamma^S(A;m) = \frac{1}{6}\ln(\mu\rho) + \tilde{\Gamma}^S(m\rho) \tag{2.6}$$

and concentrate on studying the $m\rho$ dependence of $\tilde{\Gamma}^S(m\rho)$. Then there is no loss of generality in our setting the instanton scale $\rho = 1$ henceforth.

The small-s behavior of $F(s)$, as given by the Schwinger-DeWitt expansion, reads [10]

$$s \to 0+ \quad : \quad F(s) \sim -\frac{1}{12} + \frac{1}{75}s + \frac{17}{735}s^2 - \frac{116}{2835}s^3 + \cdots \tag{2.7}$$

Insering this result into (2.4) gives rise to the following large-mass expansion of $\tilde{\Gamma}^S(m)$:

$$m \to \infty\ :\ \tilde{\Gamma}^S(m) = -\frac{1}{6}\ln m - \frac{1}{75m^2} - \frac{17}{735m^4} + \frac{232}{2835m^6} + \cdots. \tag{2.8}$$

Note that the Schwinger-DeWitt expansion is a small s expansion for $F(s)$, which naturally leads to a large m expansion for $\tilde{\Gamma}^S(m)$. To obtain an expression for $\tilde{\Gamma}^S(m)$ for general values of m we need a more general expression for $F(s)$. In the small mass limit, on the other hand, a completely independent calculation for $\tilde{\Gamma}^S(m)$ has been given [5, 10, 14], based on the fact that the massless propagators in an instanton background are known in closed-form:

$$m \to 0 \quad : \quad \tilde{\Gamma}^S(m) = \alpha\left(\frac{1}{2}\right) + \frac{1}{2}(\ln m + \gamma - \ln 2)m^2 + O(m^4), \tag{2.9}$$

where $\alpha(\frac{1}{2}) \simeq 0.145873$, and $\gamma \simeq 0.5772\ldots$ is Euler's constant. From this small m behavior it is possible to infer that $F(s) \sim -\frac{1}{4s}$ as $s \to \infty$. However, this information does not determine the magnitude of $\tilde{\Gamma}^S(m)$ even for small m, since the integral in (2.4) is dominated by contributions from non-asymptotic s-values.

In this paper we make use of the fact that the function $F(s)$ may be expressed in terms of appropriate scattering phase shifts. Note that the differential operator $-D^2$ in the instanton background (2.2) (with $\rho = 1$) can be cast in the form [5]

$$-D^2 = -\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{4}{r^2}\vec{L}^2 + \frac{4}{r^2+1}(\vec{J}^2 - \vec{L}^2) - \frac{4}{(r^2+1)^2}\vec{T}^2, \tag{2.10}$$

where $\vec{T}^2 \equiv T^a T^a$, and eigenvalue $T^2 = t(t+1)\frac{3}{4}$ appropriate to isospin $t = \frac{1}{2}$; $\vec{L}^2 \equiv L_a L_a$ with $L_a - \frac{i}{2}\eta_{\mu\nu a}x_\mu\partial_\nu$ (satisfying angular-momentum commutation relations) and eigenvalues $L^2 l(l+1)$, $l = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \cdots$; $\vec{J}^2 \equiv (\vec{L}+\vec{T})^2$ with eigenvalues $J^2 = j(j+1)$, $j = |\, l \pm t \,| = |\, l \pm \frac{1}{2}\,|$. Without any background, we have the differential operator

$$-\partial^2 = -\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{4}{r^2}\vec{L}^2, \tag{2.11}$$

which corresponds to the t=0 case of the expression (2.10). We may then consider the quantum mechanical scattering problem with the Hamiltonian $\mathcal{H} = -D^2$, viz.,

$$\mathcal{H}\Psi \equiv \left[-\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{4l(l+1)}{r^2} + \frac{4(j-l)(j+l+1)}{r^2+1} - \frac{3}{(r^2+1)^2}\right]\Psi = k^2\Psi \tag{2.12}$$

with the corresponding free Schrödinger equation given by

$$\mathcal{H}_0\Psi_0 \equiv \left[-\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{4l(l+1)}{r^2}\right]\Psi_0 = k^2\Psi_0. \tag{2.13}$$

As $r \to 0$, we assume that $\Psi,\ \Psi_0 \sim (\text{const.})\, r^{2l}$. Also, to make the spectrum discrete, it is convenient to put the system in a large spherical box of radius R, demanding a suitable boundary condition at $r = R$ (e.g., the Dirichlet condition $\psi(r = R) = 0$). Then the solutions of (2.12) and (2.13) have the asymptotic large-r behaviors

$$\Psi_{0n}(r) \ \sim\ 2Cr^{-3/2}\cos[k_0(n)(r+a)], \tag{2.14}$$

$$\Psi_n(r) \ \sim\ 2Cr^{-3/2}\cos[k_0(n)(r+a) + \eta(k(n))] \tag{2.15}$$

where $\eta(k(n))$ denotes the related scattering phase shift, and the now discrete momenta $k_0(n)$, $k(n)$ (n: nonnegative integers) satisfy the conditions [5, 16]

$$k(n+1) - k(n) = \frac{\pi}{R} + O\left(\frac{1}{R^2}\right) \ \ (= k_0(n+1) - k_0(n)),$$

$$k_0(n) = k(n) + \frac{\eta(k(n))}{R} + O\left(\frac{1}{R^2}\right). \tag{2.16}$$

This scattering mode description may be considered for every partial wave. If $[k^{l,j}(n)]^2$ and $[k_0^l(n)]^2$ denote the energy eigenvalues introduced in association with (2.12) and (2.13), respectively, the function $F(s)$ (see (1.1)) may then be represented as

$$F(s) = \sum_{l=0,\frac{1}{2},\cdots} \sum_j (2l+1)(2j+1) \sum_n \left\{ e^{-s[k^{l,j}(n)]^2} - e^{-s[k_0^l(n)]^2} \right\}, \tag{2.17}$$

including the degeneracy factor $(2l+1)(2j+1)$. The factor $(2j+1)$ corresponds to different eigenvalues of J_3, while the factor $(2l+1)$ corresponds to the eigenvalues of $\bar{L}_3$, the third component of the second set of conserved angular-momentum $\bar{L}_a \equiv -\frac{i}{2}\bar{\eta}_{\mu\nu a} x_\mu \partial_\nu$.

The phase-shift relations (2.16) imply that for large R we can write

$$e^{-s[k^{l,j}(n)]^2} - e^{-s[k_0^l(n)]^2} = e^{-s[k^{l,j}(n)]^2} \left\{ \frac{2k^{l,j}(n)\eta_{l,j}(k(n))}{R} s + O\left(\frac{1}{R^2}\right) \right\}. \tag{2.18}$$

Based on this observation, it is possible to replace the sum $\sum_n$ in (2.17) by an integral:

$$F(s) = \frac{2}{\pi} s \sum_{l=0,\frac{1}{2},\cdots} \sum_j \int_0^\infty dk e^{-k^2 s} k (2l+1)(2j+1) \eta_{l,j}(k). \tag{2.19}$$

Note that this expression is not only an infinite series but also contains an improper k-integral – hence it must be considered carefully. In the instanton background in particular, the l-sum and j-sum in (2.19) may not be considered in a completely independent way. This follows from the nature of the scattering problem as defined by (2.12) and (2.13); according to the forms for $\mathcal{H}$ and $\mathcal{H}_0$, their small-r behaviors match for a given l-value, but it is the j-value that governs the large-r behavior of the effective potential in $\mathcal{H}$, and j does not appear in $\mathcal{H}_0$. This apparent mis-match can be resolved simply by considering the phase shifts $\eta_{l,l+\frac{1}{2}}(k)$ and $\eta_{l+\frac{1}{2},l}(k)$ (with the same associated degeneracy factor) together as a package. With this understanding, the expression (2.19) can now be cast in the form

$$F(s) = \frac{2}{\pi} s \sum_{l=0,\frac{1}{2},\cdots} \int_0^\infty dk e^{-k^2 s} k (2l+1)(2l+2) \left\{ \eta_{l,l+\frac{1}{2}}(k) + \eta_{l+\frac{1}{2},l}(k) \right\}. \tag{2.20}$$

This form still requires careful treatment with regards to the l-sum and k-integration. We will argue below that the correct (gauge-invariant) procedure is to have the various terms corresponding to the same 'energy' eigenvalue (i.e., the same k^2-value) receive uniform consideration. As a convenient check on this procedure, we confirm that the predicted small-s behavior reproduces the form in (2.7).

III. SYSTEMATIC WKB PHASE-SHIFT ANALYSIS

To find the exact form of $F(s)$ with the help of (2.20), one must first have complete knowledge of the scattering phase shifts $\eta_{l,j}(k)$, and then carry out the needed infinite sum/integration in a carefully controlled way. This is not possible in general, and therefore one needs to develop a reliable approximation scheme to determine the function $F(s)$. We provide such a scheme below, which relies on a systematic WKB approximation for the scattering phase shifts in question. In contrast to previous applications [15] of the WKB method for a similar purpose (but in lower dimensions), the leading-order WKB result turns out to be insufficient even to ensure the correct value for $F(s=0)$. [Recall that from (2.7) we must have $F(s=0) = -\frac{1}{12}$ in order to construct the renormalized effective action as in (2.5)]. The necessity to include higher-order WKB contributions arises from the inaccuracy of the l-sum in (2.20) with respect to large-l contributions. While the degree of accuracy of the WKB expression for the phase shifts is generally enhanced for higher partial waves, the large degeneracy factor $(2l+1)(2l+2)$ has the consequence that leading-order WKB method is insufficient. Fortunately, as we shall see below, this can be remedied in a systematic way by including higher-order WKB correction terms.

Let us first consider the representation of $F(s)$ in the leading WKB approximation. To that end, (2.12) may be rewritten as an equation for $\bar{\Psi}(r) \equiv r^{3/2}\Psi(r)$:

$$\left\{-\frac{\partial^2}{\partial r^2} + \frac{4l(l+1)+\frac{3}{4}}{r^2} + \frac{4(j-l)(j+l+1)}{r^2+1} - \frac{3}{(r^2+1)^2}\right\}\bar{\Psi}(r) = k^2\bar{\Psi}(r). \tag{3.1}$$

For this one-dimensional Schrödinger-type equation, the leading WKB solution can be given immediately, including the usual Langer correction [19] to take into account the singular centrifugal term. Phase shifts in the leading WKB approximation, which can be extracted from this solution, read

$$\eta^{(1)}_{l,j}(k) = \int_{r_1(k)}^{\infty} dr' \sqrt{k^2 - V^{l,j}(r')} - \int_{r_0(k)}^{\infty} dr' \sqrt{k^2 - V_0^l(r')} \tag{3.2}$$

with

$$V^{l,j}(r) \equiv \frac{4(l+\frac{1}{2})^2}{r^2} + \frac{4(j-l)(j+l+1)}{r^2+1} - \frac{3}{(r^2+1)^2}, \tag{3.3}$$

$$V_0^l(r) \equiv \frac{4(l+\frac{1}{2})^2}{r^2}, \tag{3.4}$$

In (3.2), $r_1(k)$, $r_0(k)$ denote the classical turning points determined by the conditions $V^{l,j}(r_1) = k^2$ and $V_0^l(r_0) = k^2$, respectively.

We now define

$$X^{(1)}_{l,j}(s) \equiv \int_0^\infty dk\, e^{-k^2 s}\, k\, \eta^{(1)}_{l,j}(k) \tag{3.5}$$

as this kind of integral is relevant in the construction of $F(s)$ in (2.20). Using the expression (3.2) for the phase shift and changing the order of integrations, this quantity may then be rewritten as

$$X^{(1)}_{l,j}(s) = \int_0^\infty dr \left[\int_{k_1(r)}^\infty dk\, e^{-k^2 s} k \sqrt{k^2 - V^{l,j}(r)} - \int_{k_0(r)}^\infty dk\, e^{-k^2 s} k \sqrt{k^2 - V^l_0(r)} \right], \tag{3.6}$$

where $k_1(r) \equiv \sqrt{V^{l,j}(r)}$ and $k_0(r) \equiv \sqrt{V^l_0(r)}$. The k-integration in (3.6) can be carried out explicitly to give

$$X^{(1)}_{l,j}(s) = \frac{\sqrt{\pi}}{4 s^{3/2}} \int_0^\infty dr \left[e^{-sV^{l,j}(r)} - e^{-sV^l_0(r)} \right]. \tag{3.7}$$

Thus, the leading WKB expression for F(s) is:

$$\begin{aligned} F^{(1)}(s) = \frac{1}{2\sqrt{\pi}\sqrt{s}} \int_0^\infty dr \Bigg(\sum_{l=0,\frac{1}{2},\cdots} (2l+1)(2l+2) \Big\{ & e^{-sV^{l,l+\frac{1}{2}}(r)} - e^{-sV^l_0(r)} \\ & + e^{-sV^{l+\frac{1}{2},l}(r)} - e^{-sV_0^{l+\frac{1}{2}}(r)} \Big\} \Bigg). \end{aligned} \tag{3.8}$$

Higher-order WKB correction can also be included. For the Schrödinger equation (3.1), one can derive the 2nd-order and 3rd-order WKB phase shifts (i.e., $\eta^{(2)}_{l,j}(k)$ and $\eta^{(3)}_{l,j}(k)$), incorporating the Langer correction in an appropriate manner, along the line discussed in Refs. [17, 18, 19]. Leaving the somewhat involved details of this derivation elsewhere [20], we shall here only report the results for $X^{(2)}_{l,j}(s)$ and $X^{(3)}_{l,j}(s)$ (which are related to the phase shift contributions of respective order by the integral relation of (3.5)):

$$\begin{aligned} X^{(2)}_{l,j} &= \frac{\sqrt{\pi}}{4s^{3/2}} \int_0^\infty dr \Bigg[e^{-sV^{l,j}(r)} \left\{ \frac{1}{4r^2} s - \frac{1}{12} s^2 \frac{d^2 V^{l,j}(r)}{dr^2} \right\} \\ &\qquad - e^{-sV^l_0(r)} \left\{ \frac{1}{4r^2} s - \frac{1}{12} s^2 \frac{d^2 V^l_0(r)}{dr^2} \right\} \Bigg], \end{aligned} \tag{3.9}$$

$$\begin{aligned} X^{(3)}_{l,j} &= \frac{\sqrt{\pi}}{4s^{3/2}} \int_0^\infty dr \Bigg[e^{-sV^{l,j}(r)} \left\{ \frac{5s^2}{32r^4} - \frac{s^3}{48r^2} \frac{d^2 V^{l,j}(r)}{dr^2} - \frac{s^3}{288} \frac{d^4 V^{l,j}(r)}{dr^4} + \frac{7s^4}{1440} \left(\frac{d^2 V^{l,j}(r)}{dr^2} \right)^2 \right\} \\ &\qquad - e^{-sV^l_0(r)} \left\{ \frac{5s^2}{32r^4} - \frac{s^3}{48r^2} \frac{d^2 V^l_0(r)}{dr^2} - \frac{s^3}{288} \frac{d^4 V^l_0(r)}{dr^4} + \frac{7s^4}{1440} \left(\frac{d^2 V^l_0(r)}{dr^2} \right)^2 \right\} \Bigg]. \end{aligned} \tag{3.10}$$

Inserting these results into (2.20) we obtain the 3rd-order WKB expression for $F(s)$:

$$F(s)_{\text{WKB}} = F^{(1)}(s) + F^{(2)}(s) + F^{(3)}(s)$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{s}}\int_0^\infty dr \left[\sum_{l=0,\frac{1}{2},\cdots}(2l+1)(2l+2)\left\{e^{-sV^{l,l+\frac{1}{2}}(r)}H^{l,l+\frac{1}{2}}(r) - e^{-sV_0^l(r)}H_0^l(r)\right.\right.$$
$$\left.\left. +e^{-sV^{l+\frac{1}{2},l}(r)}H^{l+\frac{1}{2},l}(r) - e^{-sV_0^{l+\frac{1}{2}}(r)}H_0^{l+\frac{1}{2}}(r)\right\}\right] \qquad (3.11)$$

where the $H(r)$ functions are local functions of the potentials in (3.3) and (3.4):

$$H^{l,j}(r) = 1 + \frac{s}{4r^2} + \frac{5s^2}{32r^4} - \left(\frac{s^2}{12} + \frac{s^3}{48r^2}\right)\frac{d^2V^{l,j}(r)}{dr^2} - \frac{s^3}{288}\frac{d^4V^{l,j}(r)}{dr^4} + \frac{7s^4}{1440}\left(\frac{d^2V^{l,j}(r)}{dr^2}\right)^2, \qquad (3.12)$$

$$H_0^l(r) = 1 + \frac{s}{4r^2} + \frac{5s^2}{32r^4} - \left(\frac{s^2}{12} + \frac{s^3}{48r^2}\right)\frac{d^2V_0^l(r)}{dr^2} - \frac{s^3}{288}\frac{d^4V_0^l(r)}{dr^4} + \frac{7s^4}{1440}\left(\frac{d^2V_0^l(r)}{dr^2}\right)^2. \qquad (3.13)$$

It is important that in evaluating (3.11) the l-sum be done first, and then the r-integration. This rule regarding what operations should be done first follows [20] by adopting a definition of $F(s)$ as given by the infinite 'energy' cutoff limit (see the remark at the end of Sec. II). In fact, only with this procedure is the correct small-s behavior for $F(s)$ found (see below) – this can be taken as further (a posteriori) evidence for our rule. We have verified explicitly that carrying out the r-integration for individual partial wave contributions first and then summing leads to incorrect expressions. The difference for $F(s)$ is $\frac{1}{4s}$, which leads to a spurious quadratic divergence in the s-integral.

Based on the form (3.11), one can determine the function $F(s)_{\mathrm{WKB}}$ numerically. The infinite l-sum in this formula can be handled using the Euler-Maclaurin summation formula [21] (some care is needed for small-r values), and the result is a rapidly convergent series. Numerical integration with the resulting function of r can then be performed with very high accuracy. Figure 1 shows a comparison of the WKB approximations for $F(s)$ with one, two and three terms in the WKB expansion. We plot the function $F(s)_{\mathrm{WKB}}$ (i.e., including up to 3rd-order WKB terms), together with the corresponding plots for $F_1(s) \equiv F^{(1)}(s)$ (i.e., the result based on the leading WKB phase-shift expressions only) and $F_2(s) \equiv F^{(1)}(s) + F^{(2)}(s)$ (i.e., including up to 2nd-order WKB corrections). The value we found for $F_1(s)$ at $s = 0$ is $-\frac{1}{24}$, while the correct value is $-\frac{1}{12}$ (see (2.7)). Both 2nd and 3rd order WKB give the correct value for $F(0)$. Thus, as remarked earlier, the leading WKB result alone is not sufficient here even for the renormalization discussion. With the 2nd order WKB expression, $F_2(s)$, there is

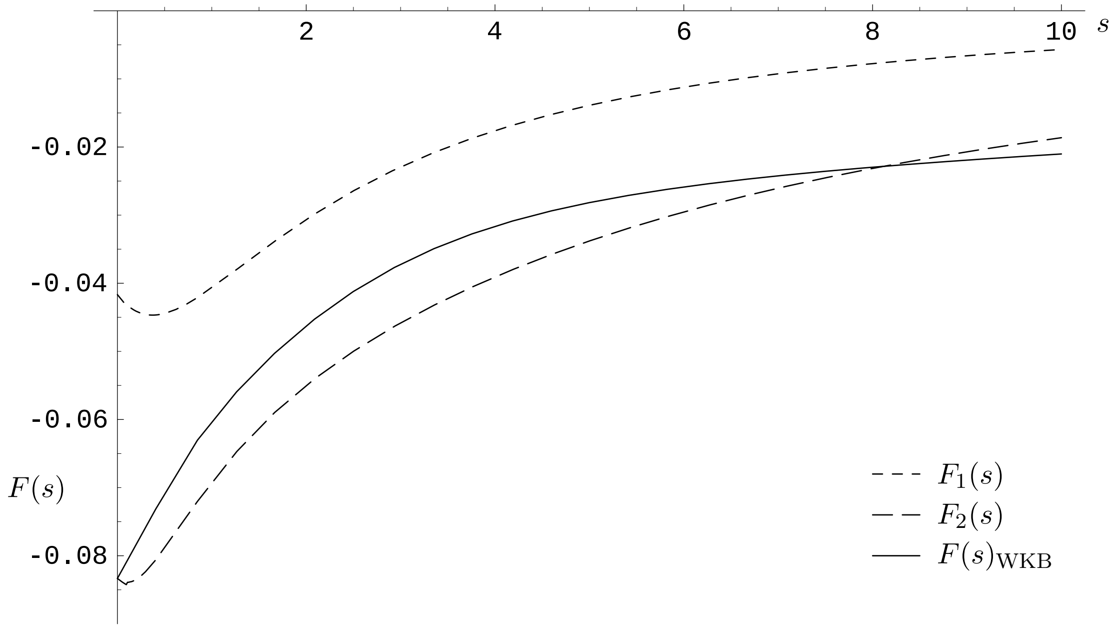

FIG. 1: Plot of $F(s)$, showing the 1st, 2nd and 3rd order WKB approximations.

no problem with renormalization, but, for its first derivative at $s = 0$, we found $F_2'(0) = -\frac{1}{90}$, which does not agree with the small s result from the Schwinger-DeWitt expansion in (2.7). On the other hand, going to 3rd order WKB, $F(s)_{\rm WKB}$ gives the correct value for this first derivative, with $F'(0)_{\rm WKB} = \frac{1}{75}$. With the small-$s$ behavior satisfactorily taken care of, we expect good agreement with the large mass behavior of the effective action, as can be confirmed from Figure 2. For very large s (which corresponds to the $m \to 0$ limit), the 3rd order WKB expression $F(s)_{\rm WKB}$ approaches zero faster than the true $F(s)$. Nevertheless, we show below that this 3rd order expression gives an excellent approximation even in the extreme massless limit.

Our function $F(s)_{\rm WKB}$ can be used to determine the one-loop effective action for arbitrary mass value. We insert $F(s)_{\rm WKB}$ into (2.4), integrate over the proper-time s (numerically), and renormalize according to (2.5). Extracting $\tilde{\Gamma}^S(m)$, as defined in (2.6), we obtain the plot shown in Fig. 2. Also shown in Fig. 2 are the curves based on the inverse mass expansion (2.8) and small mass expansion (2.9). Evidently, our WKB-based plot corresponds to a smooth interpolation of the latter two curves, as conjectured in Ref.[10] earlier. The agreement at large m is excellent. For $m = 0$, our WKB-based prediction gives rise to the value 0.137827, which is about 6% off from 't Hooft's exact value, $\alpha(\frac{1}{2}) \simeq 0.145873$.

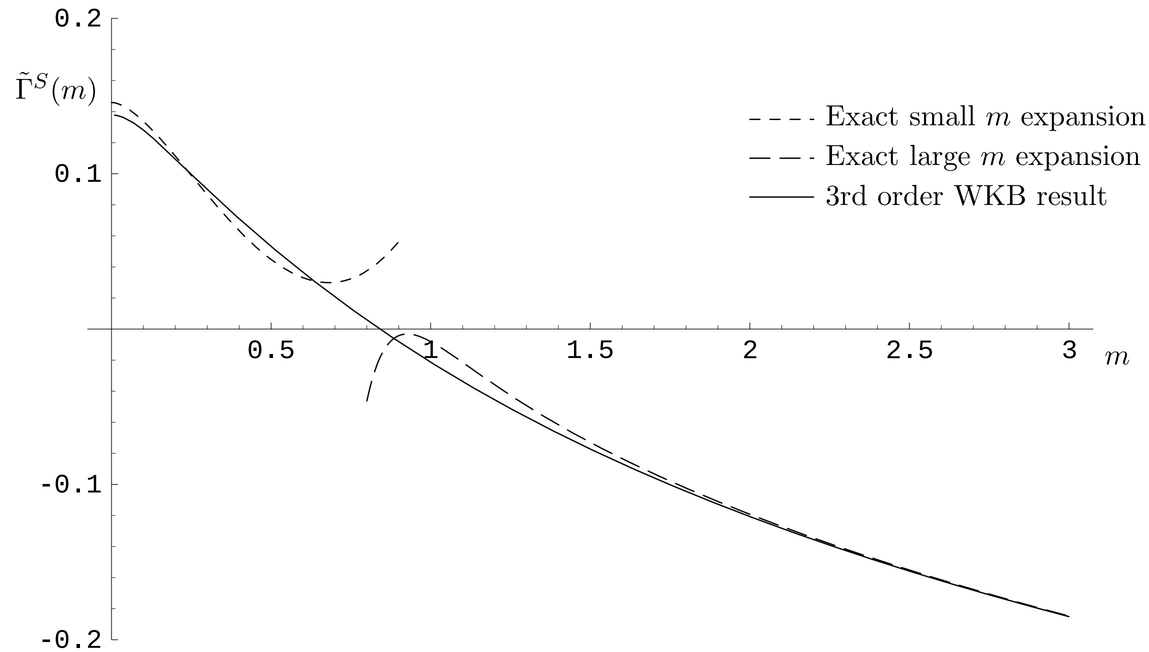

FIG. 2: Plot of $\tilde{\Gamma}^S(m)$, comparing the 3rd order WKB result with the exact extreme large and small mass limits.

[If $F_2(s)$ were used instead of $F(s)_{\rm WKB}$ for this calculation, the predicted value for $\tilde{\Gamma}^S(0)$ would be 0.158084, which is about 10% off from the exact value]. The discrepancy from the exact result is expected to be largest for $m = 0$; hence, we estimate that our WKB-based prediction of $\tilde{\Gamma}^S(m)$, for arbitrary mass m, is good to 6% accuracy.

IV. FIELD-THEORETIC DERIVATIVE EXPANSION APPROACH

The field-theoretic derivative expansion provides a quick and extremely simple estimate of the one-loop effective action in an instanton background for any value of the quark mass m, and we show here that even the leading order term gives surprisingly good agreement. The philosophy of the derivative expansion is to compute the one-loop effective Lagrangian for a covariantly constant background field, which can be done exactly, and then perturb around this constant background solution. The leading order derivative expansion approximation for the effective action is obtained by first taking the (exact) expression for the effective Lagrangian in a covariantly constant background, substituting the space-time dependent background, and then integrating over space-time.

For an instanton background, which is self-dual, we should base our derivative expansion

approximation on a covariantly constant and self-dual background: one may then set (in a suitable gauge)[22] $F_{\mu\nu} = F^{\rm AB}_{\mu\nu} n^a\, T^a$, where the abelian field strength $F^{\rm AB}_{\mu\nu}$ is self-dual, and n^a is a unit vector in color-space. Comparing with the large ρ limit of the $su(2)$ instanton field strength (2.3), we identify $F^{\rm AB}_{\mu\nu} n^a = -\frac{4}{\rho^2}\eta_{\mu\nu a}$. The exact one-loop scalar effective Lagrangian for a covariantly constant self-dual field is well-known (see, e.g., Eq. (2.11) in [23]). Substituting the instanton form we obtain the derivative expansion (DE) approximation (here we set $\rho = 1$, as before)

$$\mathcal{L}^{\rm scalar}_{\rm DE} = -\frac{2}{(4\pi)^2}\int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s}\left[\left(\frac{\frac{\sqrt{12}\,s}{(1+r^2)^2}}{\sinh\left(\frac{\sqrt{12}\,s}{(1+r^2)^2}\right)}\right)^2 - 1 + \frac{1}{3}\left(\frac{\sqrt{12}\,s}{(1+r^2)^2}\right)^2\right] \tag{4.1}$$

The leading derivative expansion for the effective action is then obtained by integrating the effective Lagrangian (4.1) over space-time. Note that (4.1) has been renormalized on-shell, so that $\mu = m$ in (2.6). To study both the large and small mass limits it is useful to express (4.1) in a different, but equivalent, form using the identity:

$$\int_0^\infty \frac{du}{u^3} e^{-2\kappa u}\left[\left(\frac{u}{\sinh u}\right)^2 - 1 + \frac{u^2}{3}\right] = -\frac{1}{3}\ln\kappa + 4\int_0^\infty \frac{dx\, x}{e^{2\pi x}-1}\,\ln\left(x^2+\kappa^2\right) \tag{4.2}$$

Then the large mass expansion of the effective action is straightforward:

$$\begin{aligned}\Gamma^S_{\rm DE}(A;m) &= 2\pi^2\int_0^\infty r^3\, dr\,\mathcal{L}^{\rm scalar}_{\rm DE}\\ &= -6\int_0^\infty \frac{r^2\, d(r^2)}{(1+r^2)^4}\int_0^\infty \frac{dx\, x}{e^{2\pi x}-1}\,\ln\left(1+\frac{48x^2}{m^4(1+r^2)^4}\right)\\ &\sim 3\sum_{l=1}^\infty \frac{48^l\,\mathcal{B}_{2l+2}}{(2l+2)(2l+1)(2l)(4l+3)}\frac{1}{m^{4l}} \quad , \quad m\to\infty \end{aligned} \tag{4.3}$$

where $\mathcal{B}_l$ are the Bernoulli numbers. Note that the large mass expansion (4.3) of the leading derivative expansion approximation begins with $\frac{1}{m^4}$, rather than the true $\frac{1}{m^2}$ behavior, because for the covariantly constant self-dual field the corresponding Schwinger-DeWitt coefficient vanishes by group theoretic traces.

To study the small mass expansion, we perform the r integral in (4.3) to obtain an exact integral representation of the leading derivative expansion approximation

$$\begin{aligned}\Gamma^S_{\rm DE}(A;m) \;=\; -\frac{1}{14}\int_0^\infty \frac{dx\, x}{e^{2\pi x}-1}\Bigg\{&-84 + 14\ln\left(1+\frac{48x^2}{m^4}\right) + 7\sqrt{3}\,\frac{m^2}{x}\arctan\left(\frac{4\sqrt{3}\,x}{m^2}\right)\\ &+768\frac{x^2}{m^4}\;{}_2F_1\left(1,\frac{7}{4},\frac{11}{4};-\frac{48x^2}{m^4}\right)\Bigg\}\end{aligned} \tag{4.4}$$

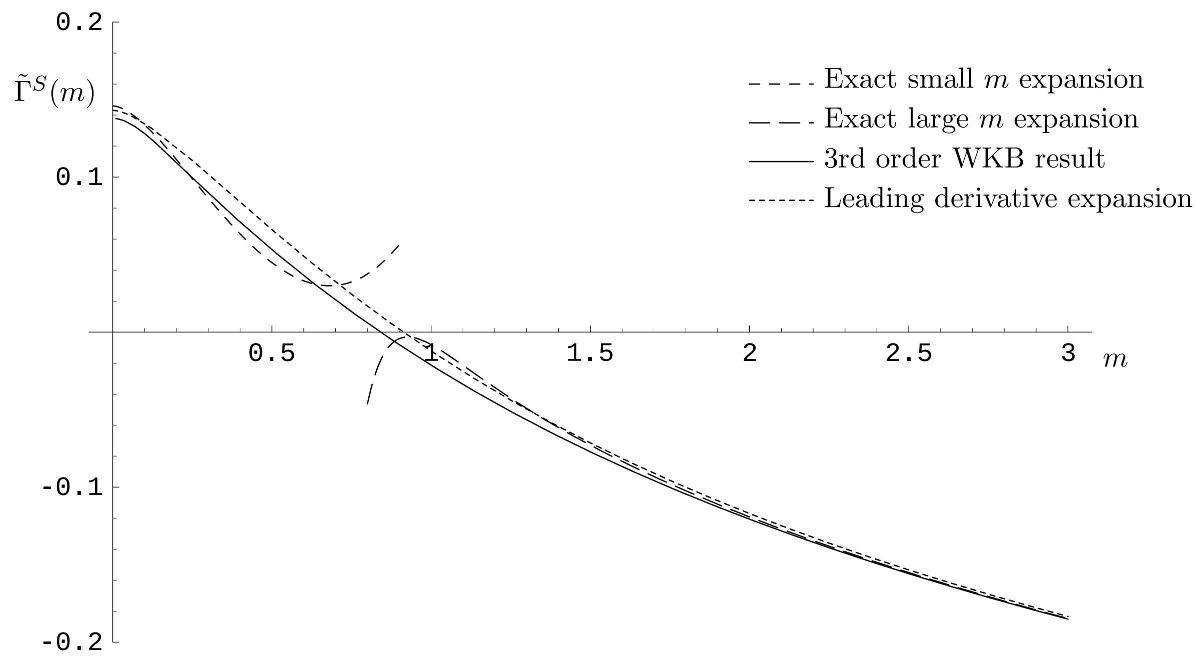

FIG. 3: Plot of $\tilde{\Gamma}^S(m)$, comparing the leading derivative expansion approximation with the 3rd order WKB result and with the exact extreme large and small mass limits.

It is now simple to expand each term for small mass to obtain the leading behavior

$$\Gamma^S_{\rm DE}(A;m) \sim \frac{1}{6}\ln(m) + \left(\frac{5}{36} - \frac{1}{24}\ln(48) - \zeta'(-1)\right) + \frac{\sqrt{3}}{4} m^2 \ln(m) + \dots\ , \quad m \to 0 \quad (4.5)$$

The agreement with the leading small mass behavior (2.9) of the exact result is quite remarkable. The coefficient $\frac{1}{6}$ of the $\ln(m)$ term agrees, as it must by virtue of the β-function. The constant term $\left(\frac{5}{36} - \frac{1}{24}\ln(48) - \zeta'(-1)\right) \simeq 0.14301$ is only 2% away from 't Hooft's value of $\alpha(\frac{1}{2}) \simeq 0.14587$, and the coefficient of the $m^2 \ln(m)$ term is $\frac{\sqrt{3}}{4} \simeq 0.433$, compared to Carlitz and Creamer's result of 0.5. Figure 3 shows a comparison of the associated function $\tilde{\Gamma}^S(m)$ with the WKB result of the previous section and the exact large and small mass limits. The agreement is surprisingly good for such a crude approximation [24].

V. CONCLUSION

In this letter we have presented a computation of the fermion determinant in an instanton background for all values of the quark mass. Using 3rd order WKB approximation for the scattering phase shifts we obtained a result which interpolates very well between the known extreme large and small mass results. We expect even higher order WKB terms

to improve the accuracy further. Also, the agreement in the small mass regime might be improved by determining the large s asymptotic behavior of the proper-time function $F(s)$ generalizing the method of Barzinsky and Mukhanov [25]. Finally, we showed that a very crude leading order derivative expansion approximation based on a covariantly constant self-dual background leads to surprisingly good agreement.

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